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A volume formula for asymptotic hyperbolic tetrahedra with an application to quantum field theory

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ABSTRACT

In this paper, a volume formula for hyperbolic tetrahedra with one vertex at infinity is derived. This yields a new representation of the four-dimensional three-point function of Euclidean quantum field theory, i.e., of $\int_{\mathbb{R}^4} \prod_{k=1}^{3} (q_1^2 + \cdots + q_4^2 + b_{1k} q_1 + \cdots + b_{4k} q_4 + d_k)^{-1} \, \mathrm{d}q_1 \cdots \mathrm{d}q_4$, through 15 dilogarithms containing inverse trigonometric functions which are symmetric in the parameters b_{jk} and d_k, $j = 1, \ldots, 4$, $k = 1, 2, 3$.

1. INTRODUCTION AND NOTATIONS

In contrast to Euclidean space, a triangle in *hyperbolic space* is determined – up to congruence – by its inner angles $\alpha_1, \alpha_2, \alpha_3$. Their sum must be smaller than π and the excess coincides with the area F of this triangle:

$$F = \pi - \alpha_1 - \alpha_2 - \alpha_3.$$

Similarly, a tetrahedron T in hyperbolic space is determined by its six *dihedral angles* $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$, i.e. the angles between the four faces of this tetrahedron. (Propositions 1 and 2 state complete sets of inequalities these angles have to fulfil, thereby answering a question of W. Fenchel, comp. Remark 1 to Theorem 3.) In order to calculate the volume V of T, one can dissect T into six *orthoschemes*, i.e. tetrahedra satisfying, after a suitable reordering of the vertices, $\alpha_{13} = \alpha_{14} = \alpha_{24} = \pi/2$. Lobachevsky's formula for the volume V of such an orthoscheme (cf. [13, § 30, (160), p. 97]) reads

$$V = \frac{1}{4}\Big[Л(\delta+\alpha_{12}) + Л(\delta-\alpha_{12}) + Л(\delta+\alpha_{34}) + Л(\delta-\alpha_{34}) - Л\Big(\delta+\alpha_{23}+\frac{\pi}{2}\Big) - Л\Big(\delta-\alpha_{23}+\frac{\pi}{2}\Big) - 2Л\Big(\delta+\frac{\pi}{2}\Big)\Big],$$

where $Л(x) := -\int_0^x \ln(2|\sin\theta|)\,d\theta$ and

$$\cos(\delta) = \frac{\cos(\alpha_{12})\cos(\alpha_{34})}{\sqrt{\cos^2(\alpha_{12})+\cos^2(\alpha_{34})-\sin^2(\alpha_{23})}}.$$

There have been given new proofs and extensions of this famous formula later on by several authors, see, e.g., [5, (5.4), p. 23], [3, 3., pp. 300–303; in particular (3.16), p. 302], [4, 5.8, (18), p. 251], [6, IX.4, (17), p. 219], [8, Theorems II and III, pp. 562 and 565], [9, Theorem 14.5, (14.48), p. 319].

A considerable drawback of the dissection method is that the resulting expression for the volume of an arbitrary tetrahedron T consists of 42 dilogarithms which contain the dihedral angles in an unsymmetric way, comp. [16, Proposition 8 and Remark 2 to Proposition 8, pp. 106, 108]. In fact, there are many different ways to dissect T. Lobachevsky seems to have been aware already of the corresponding problem (cf. [13, p. vi, next to last paragraph]), namely of finding a volume formula for tetrahedra in hyperbolic space that is *symmetric* in the dihedral angles. This problem is, to the best of my knowledge, still open.

In this paper, I shall consider the case of an *asymptotic* tetrahedron T, i.e. one with at least one vertex at infinity. After a suitable reordering of the vertices, this means that $\alpha_{12}+\alpha_{13}+\alpha_{23}=\pi$. The volume V of T is then given as a sum of 15 dilogarithms which contain *linear* functions of the dihedral angles, cf. Theorem 3 and formula (10). (The dissection method yields instead a sum of 18 dilogarithms which contain the dihedral angles in a nonlinear fashion, cf. [16, Remark 1 to Proposition 8, p. 108]. Equating that with formula (10) amounts to one of the many identities between dilogarithmic functions, comp. [11], [12].) By this we extend J. Milnor's formula (cf. [14, Lemma 2, p. 18])

$$V = Л(\alpha_{12}) + Л(\alpha_{13}) + Л(\alpha_{23}),$$

which refers to an *ideal* tetrahedron, i.e. one with *all* vertices at infinity, comp. Remark 2 to Theorem 3.

This paper was motivated by the article [16], in which the scalar m-point function in n-dimensional Euclidean space, i.e. the integral

$$(1)\qquad \int_{\mathbb{R}^n} \prod_{k=1}^{m} (q_1^2+\cdots+q_n^2+b_{1k}q_1+\cdots+b_{nk}q_n+d_k)^{-1}\,dq_1\cdots dq_n,$$

is first reduced to a sum over Gaussian integrals of the type

$$(2)\qquad \int_{\mathbb{R}_+^n} \exp\Big(-\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_ix_j\Big)\,dx_1\cdots dx_n$$

and, second, these latter integrals are represented explicitly in terms of logarithmic and inverse trigonometric functions in the case of $n = 2$ and of $n = 3$, and by sums of 42 Clausen functions in the physically relevant case of $n = 4$, respectively. The volume formula for asymptotic tetrahedra yields a new, simple, and symmetric expression for (2) by 15 Clausen functions in the case of $n = 4$, $c_{44} = 0$ (cf. (10), Theorem 2). From the physical viewpoint, this case corresponds to the three-point function in Euclidean $\mathbb{R}^4$, i.e., to setting $m = 3$ and $n = 4$ in (1) (cf. Theorem 4).

Let me mention that the four-point function – that is the case of $m = n = 4$ in (1) – can be expressed as a difference of two three-point functions by a procedure going back to Lobachevsky, cf. [13, § 33, pp. 99, 100]. Geometrically, this is done by extending one of the edges of a hyperbolic tetrahedron to infinity and thus representing its volume by the difference of the volumes of two asymptotic tetrahedra with one common vertex at infinity. Referring to the integral (2), this amounts to performing the substitution $x_3' = x_3 + ux_4$ where $u := c_{34} - \sqrt{c_{34}^2 - c_{33}c_{44}}$ (see also [15, pp. 431, 432]). The resulting representation of the integral in (2) for $n = 4$ by 24 Clausen functions (6 of the 30 cancel) is not stated in this paper, since it is again – as the one in [16, Proposition 8, p. 106] – unsymmetric in the parameters c_{ij}. As discussed already above, it is still an open problem to express the four-point function in a symmetric way through the parameters b_{jk} and d_k, which represent the momenta and masses of the particles involved.

Let us establish some notations. For real numbers $a < b$ or $b = \infty$, the open and closed intervals from a to b are denoted by $]a, b[$ and $[a, b]$, respectively. (Note that brackets will also be used in Section 3 for the Lorentz product.) We consider $\mathbb{R}^n$ as a Euclidean space with the inner product $\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n$ and write $|x| := \sqrt{\langle x, x\rangle}$. The unit sphere $\{\omega \in \mathbb{R}^n : |\omega| = 1\}$ in $\mathbb{R}^n$ is denoted by $\mathbb{S}_{n-1}$ – the subscript indicating its dimension as a manifold – and $d\sigma(\omega)$ denotes the surface measure on $\mathbb{S}_{n-1}$. The set of non-negative real numbers $\{x \in \mathbb{R} : x \geq 0\}$ is abbreviated by $\mathbb{R}_+$, and $\mathbb{R}_+^n := \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n}$. We write I for the unit matrix and A^{ad} for the adjoint matrix of the matrix A, i.e., A_{ij}^{ad} is $(-1)^{i+j}$ times the determinant of the matrix resulting from A after deletion of the i-th column and of the j-th row. (Hence $AA^{\mathrm{ad}} = A^{\mathrm{ad}}A = \det(A)I$.) gl_n denotes the set of all real-valued n-by-n matrices. arccos stands for the principal value of this function, i.e. that which has its range in the interval $[0, \pi]$.

2. SCHLÄFLI'S FORMULA

For completeness, we repeat first, in a slightly generalized setting, H. Kneser's proof of Schläfli's formula ([10], [4, 5.1, Satz 1, p. 198]). In geometric language, this formula (see (3) below) expresses the differential of the volume of an $(n-1)$-dimensional spherical or hyperbolic simplex through the differentials of its dihedral angles multiplied by the volumes of the $(n-3)$-dimensional subsimplices associated with them ([18, Satz, p. 235]). We shall state and prove instead Schläfli's formula in a purely analytic way, but we will come back to its geometric content in the next section.

Theorem 1. *Let $n \in \mathbb{N}$ and fix $a_1, \ldots, a_n \in \mathbb{R}$. For*

$$t = (t_{12}, \ldots, t_{1n}, t_{23}, \ldots, t_{2n}, t_{34}, \ldots, t_{n-1,n}) \in \mathbb{R}^{n(n-1)/2},$$

define $A(t) \in \mathrm{gl}_n$ by

$$A(t) := (a_{ij})_{i,j=1}^n, \quad a_{ij} := \begin{cases} a_i : i = j, \\ t_{ij} : i < j, \\ t_{ji} : i > j. \end{cases}$$

For $C \in \mathrm{gl}_n$ and $1 \le i < j \le n$, let $C_{[ij]} \in \mathrm{gl}_{n-2}$ denote the matrix resulting from C after deletion of the i-th and j-th rows and columns. If, furthermore,

$$N_n := \{C \in \mathrm{gl}_n : \forall x \in \mathbb{R}_+^n \setminus \{0\} : \langle Cx, x\rangle > 0\}, \quad N_0 := \{0\},$$

$$M_n := \{t \in \mathbb{R}^{n(n-1)/2} : \det A(t) \neq 0, A(t)^{-1} \in N_n\},$$

$$f_n : N_n \longrightarrow \mathbb{R} : C \longmapsto \sqrt{|\det C|} \int_{\mathbb{R}_+^n} \mathrm{e}^{-\langle Cx, x\rangle}\,\mathrm{d}x, \quad f_0(0) := 1,$$

$$g_n : M_n \longrightarrow N_n : t \longmapsto A(t)^{-1},$$

then, for $n \ge 2$,
(a) *M_n is open in $\mathbb{R}^{n(n-1)/2}$ and $f_n \circ g_n$ is in $C^\infty(M_n)$;*
(b) *for $t \in M_n$, $1 \le i < j \le n$ and $a_i a_j \neq t_{ij}^2$, we have*

$$\frac{\partial(f_n \circ g_n)}{\partial t_{ij}}(t) = \frac{f_{n-2}(g_n(t)_{[ij]})}{2\sqrt{|a_i a_j - t_{ij}^2|}}. \tag{3}$$

Proof. (a) A matrix $C \in \mathrm{gl}_n$ belongs to N_n if and only if

$$\min\{\langle C\omega, \omega\rangle : \omega \in \mathbb{R}_+^n \cap \mathbb{S}_{n-1}\}$$

is positive. Since this minimum is a continuous function of C, it follows that N_n and hence M_n are open sets. If $\epsilon > 0$ is a lower bound for this minimum on the compact set $K \subset N_n$, then every derivative of the function $C \mapsto \exp(-\langle Cx, x\rangle)$ is, independently of $x \in \mathbb{R}_+^n$ and of $C \in K$, dominated by some multiple of $\exp(-\epsilon|x|^2/2) \in L^1(\mathbb{R}_+^n)$. This implies that $f_n \circ g_n$ is a smooth function.
(b) If $t \in N_n$ and $C := (c_{ij})_{i,j=1}^n := A(t)^{-1}$, then, for $1 \le i < j \le n$, we have

$$\begin{aligned} \frac{\partial\sqrt{|\det C|}}{\partial t_{ij}} &= -\frac{1}{2}\frac{\operatorname{sign}(\det A(t))}{|\det A(t)|^{3/2}} \cdot \frac{\partial \det A(t)}{\partial t_{ij}} \\ &= -\frac{1}{2}\frac{\operatorname{sign}(\det A(t))}{|\det A(t)|^{3/2}} \cdot 2A(t)_{ij}^{\mathrm{ad}} \\ &= -\frac{c_{ij}}{\sqrt{|\det A(t)|}} = -c_{ij}\sqrt{|\det C|}. \end{aligned}$$

Furthermore,

$$\frac{\partial\,\mathrm{e}^{-\langle Cx, x\rangle}}{\partial t_{ij}} = -\sum_{k=1}^n \sum_{l=1}^n \mathrm{e}^{-\langle Cx, x\rangle} x_k x_l \frac{\partial c_{kl}}{\partial t_{ij}}.$$

From

$$0 = \frac{\partial I}{\partial t_{ij}} = \frac{\partial}{\partial t_{ij}} (C \cdot A(t)) = \frac{\partial C}{\partial t_{ij}} A(t) + C \frac{\partial A(t)}{\partial t_{ij}}$$

we infer that $\partial c_{kl}/\partial t_{ij} = -c_{ki} c_{jl} - c_{kj} c_{il}$ and hence, taking into account that C is symmetric,

$$\begin{aligned}\frac{\partial \int_{\mathbb{R}^n_+} e^{-\langle Cx, x\rangle} \, dx}{\partial t_{ij}} &= \frac{1}{2} \int_{\mathbb{R}^n_+} \frac{\partial \langle Cx, x\rangle}{\partial x_i} \frac{\partial \langle Cx, x\rangle}{\partial x_j} e^{-\langle Cx, x\rangle} \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^n_+} \frac{\partial^2 e^{-\langle Cx, x\rangle}}{\partial x_i \partial x_j} \, dx + c_{ij} \int_{\mathbb{R}^n_+} e^{-\langle Cx, x\rangle} \, dx.\end{aligned}$$

Finally, by integration with respect to x_i and x_j and since $\det C_{[ij]} = (a_i a_j - t_{ij}^2) \det C \neq 0$ (cf. [7, Chapter I, § 4, (33), p. 21]), we obtain

$$\begin{aligned}\frac{\partial (f_n \circ g_n)}{\partial t_{ij}}(t) &= \frac{1}{2} \frac{\sqrt{|\det C|}}{\sqrt{|\det C_{[ij]}|}} f_{n-2}(C_{[ij]}) \\ &= \frac{f_{n-2}(g_n(t)_{[ij]})}{2\sqrt{|a_i a_j - t_{ij}^2|}}. \qquad \square\end{aligned}$$

Let us observe that M_n is the disjoint union of the open subsets of $t \in \mathbb{R}^{n(n-1)/2}$, where, besides the conditions $\det A(t) \neq 0$ and $C := A(t)^{-1} \in N_n$, also the signature of the quadratic form $\langle Cx, x\rangle$, i.e. the number of positive eigenvalues of $A(t)$ (counted with multiplicity) is fixed. In the case of the integrals in (1), $\langle Cx, x\rangle$ in (2) defines a Lorentz metric, i.e., C has exactly one positive eigenvalue (cf. [16, Proposition 4, p. 95]), and, therefore, we pay attention to this case only. We shall also weaken the condition $A(t)^{-1} \in N_n$ and require instead that the integral $\int_{\mathbb{R}^n_+} \exp(-\langle Cx, x\rangle) \, dx$ is convergent. Next, the set P_n corresponding in this way to M_n is described.

Proposition 1. *Let* $2 < n \in \mathbb{N}$ *and for* $\alpha = (\alpha_{12}, \ldots, \alpha_{1n}, \alpha_{23}, \ldots, \alpha_{n-1,n}) \in \mathbb{R}^{n(n-1)/2}$, *set* $\alpha_{ij} := \alpha_{ji}$ *if* $i > j$ *and let* $\tilde{A}(\alpha) \in \mathrm{gl}_n$ *be defined by*

$$\tilde{A}(\alpha) := (a_{ij})_{i,j=1}^n, \quad a_{ij} := \begin{cases} -1 : & i = j, \\ \cos(\alpha_{ij}) : & i \neq j. \end{cases}$$

For $d \in \mathbb{R}^n$, *denote by* $\mathrm{diag}(d)$ *the diagonal matrix with the elements* d_i, $i = 1, \ldots, n$, *along the diagonal and, furthermore, set*

$P_n := \{\alpha = (\alpha_{12}, \ldots, \alpha_{1n}, \alpha_{23}, \ldots, \alpha_{n-1,n}) \in [0, \pi]^{n(n-1)/2}$:

(i) $\tilde{A}(\alpha)$ *has* 1 *positive and* $n-1$ *negative eigenvalues* (*with multiplicity*);

(ii) $\forall i, j \in \{1, \ldots, n\}$ *with* $i \neq j$: $(-1)^n \tilde{A}(\alpha)^{\mathrm{ad}}_{ii} \leq 0, (-1)^n \tilde{A}(\alpha)^{\mathrm{ad}}_{ij} < 0\}$,

$Q_n := \{C \in \mathrm{gl}_n :$

(i) *C is symmetric*;

(ii) *C has* 1 *positive and $n-1$ negative eigenvalues* (*with multiplicity*);

(iii) $\int_{\mathbb{R}_+^n} \mathrm{e}^{-\langle Cx,x\rangle}\,\mathrm{d}x$ *converges*$\}$.

Then the following holds:

(a) *For* $(c_{ij})_{i,j=1}^n \in Q_n$, $2 \le r \le n$, *and* $1 \le i_1 < \cdots < i_r \le n$,

(4) $$(-1)^r \det((c_{i_k i_l})_{k,l=1}^r) < 0;$$

(b) *condition* (iii) *in the definition of Q_n can be replaced by*

$$(\mathrm{iii})' \;\; \forall i,j \in \{1,\dots,n\} \text{ with } i \ne j : c_{ii} \ge 0, c_{ij} > 0;$$

(c) *the map*

$$h_n : P_n \times]0,\infty[^n \longrightarrow Q_n : (\alpha, d) \longmapsto \mathrm{diag}(d)\tilde{A}(\alpha)^{-1}\mathrm{diag}(d)$$

is well-defined and bijective.

Proof. (a) Let $C := (c_{ij})_{i,j=1}^n \in Q_n$. From condition (iii), we infer that $c_{ii} \ge 0$, $i = 1,\dots,n$. (Otherwise the integral with respect to x_i diverges for all fixed $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$, and Fubini's theorem yields a contradiction to condition (iii).) Therefore, if $\{i_1,\dots,i_n\} = \{1,\dots,n\}$, and if there is no zero in the sequence

(5) $$1, c_{i_1 i_1}, c_{i_1 i_1} c_{i_2 i_2} - c_{i_1 i_2}^2, \det((c_{i_k i_l})_{k,l=1}^3), \dots, C_{i_n i_n}^{\mathrm{ad}}, \det C,$$

then condition (ii) in the definition of Q_n implies, by Jacobi's theorem (cf. [7, Chapter X, § 3, p. 303]), that this sequence has exactly $n-1$ variations of sign and hence (4) holds. If some of the members of this sequence vanish, by successively changing $c_{i_r i_r}$, $r = 1,\dots,n-1$, in such a way that conditions (i), (ii) in the definition of Q_n remain valid, $c_{i_1 i_1}$ is positive, the signs of all non-zero sequence members are unchanged, and no zero appears in the sequence, we again obtain a matrix satisfying (4). Thus the original matrix fulfills

$$(-1)^r \det((c_{i_k i_l})_{k,l=1}^r) \le 0, \quad r = 1,\dots,n.$$

We shall show that this inequality is strict for $r \ge 2$ and this will prove statement (a). In fact, if $c_{i_1 i_1} c_{i_2 i_2} - c_{i_1 i_2}^2 = 0$, we could slightly increase $c_{i_1 i_1}$ and $c_{i_2 i_2}$ to obtain $c_{i_1 i_1} > 0$, $c_{i_1 i_1} c_{i_2 i_2} - c_{i_1 i_2}^2 > 0$ and thereafter slightly increase the other $c_{i_r i_r}$ such that (i), (ii) in the definition of Q_n remain valid and that there is no zero in (5). This would yield less than $n-1$ variations of sign which is a contradiction. Similarly, if $r > 2$, $\det((c_{i_k i_l})_{k,l=1}^r) = 0$ and $\det((c_{i_k i_l})_{k,l=1}^{r-1}) \ne 0$, then any increase of $c_{i_r i_r}$ would give the same sign to these two determinants and a slight further changement would produce a sequence with $c_{i_1 i_1} > 0$, no zeros and too few variations of sign. This yields a contradiction and hence (4) holds.

(b) Due to (4), $|c_{ij}| > \sqrt{c_{ii} c_{jj}}$ for $C \in Q_n$ and $1 \le i < j \le n$. But $c_{ij} < -\sqrt{c_{ii} c_{jj}}$ is impossible since then $\int_{\mathbb{R}_+^n} \exp(-\langle Cx,x\rangle)\,\mathrm{d}x$ would diverge (again by

Fubini's theorem). Hence $c_{ij} > 0$ and condition (iii)′ holds for $C \in Q_n$. Conversely, if C fulfills (iii)′, the introduction of polar co-ordinates $x = r\omega$ yields

$$\int_{\mathbb{R}^n_+} e^{-\langle Cx, x\rangle}\, dx = \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \int_{\mathbb{R}^n_+ \cap \mathbb{S}_{n-1}} \frac{d\sigma(\omega)}{\langle C\omega, \omega\rangle^{n/2}}.$$

The denominator of the fraction in the last integral can vanish on the axes only. Since, e.g., at the point $(0, \dots, 0, 1)$ and in the local co-ordinates $\omega' = (\omega_1, \dots, \omega_{n-1})$, the estimate $\langle C\omega, \omega\rangle \geq \epsilon|\omega'|$ holds for some $\epsilon > 0$ and all $\omega' \in \mathbb{R}^{n-1}_+$ with $|\omega'| < \epsilon$, this last integral is clearly convergent. (Here $n > 2$ is used.)

(c) Part (b) implies that h_n is well-defined. The injectivity being obvious, let us finally show that h_n is surjective. If $C := (c_{ij})^n_{i,j=1} \in Q_n$, then, by (4), $C^{\mathrm{ad}}_{ii} \neq 0$, and we can set $d_i := 1/\sqrt{|(C^{-1})_{ii}|}$, $i = 1, \dots, n$, and determine $\alpha \in [0, \pi]^{n(n-1)/2}$ from $\tilde{A}(\alpha) = \operatorname{diag}(d)C^{-1}\operatorname{diag}(d)$. Note that

$$(C^{-1})_{i_1 i_1}(C^{-1})_{i_2 i_2} - (C^{-1})^2_{i_1 i_2} = \det((c_{i_k i_l})^n_{k,l=3})/\det C \geq 0$$

if $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ ([7, Chapter I, § 4, (33), p. 21]) and hence $|(C^{-1})_{ij}| d_i d_j \leq 1$. (For $n > 3$ and $i \neq j$, this inequality is sharp and thus $\alpha \in]0, \pi[^{n(n-1)/2}$.) This shows that h_n is surjective and completes the proof. □

Remarks. 1. Note that the composite mapping

$$f_n \circ h_n(\alpha, d) = f_n(\operatorname{diag}(d)\tilde{A}(\alpha)^{-1}\operatorname{diag}(d)) = f_n(\tilde{A}(\alpha)^{-1})$$

is independent of d. (By abuse of notation, we extend the definition of f_n to Q_n.)

2. From statement (b), we infer especially that $\mathring{Q}_n$, the interior of Q_n in the set of all symmetric matrices, equals $Q_n \cap N_n$ and that $C \in \mathring{Q}_n$ iff $C \in Q_n$ and $c_{ii} > 0$, $i = 1, \dots, n$. Similarly, $\mathring{P}_n$, the interior of P_n in $\mathbb{R}^{n(n-1)/2}$, is the set of all $\alpha \in P_n$ for which both inequalities in condition (ii) are sharp.

Since we are mostly interested in the case of $n = 4$, we next provide more explicit descriptions of P_4 and of $\mathring{P}_4$.

Proposition 2. *For* $\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) \in \mathbb{R}^6$, *set* $\alpha_{ij} := \alpha_{ji}$ *if* $i > j$. *For pairwise different* $i, j, k \in \{1, 2, 3, 4\}$, *define, whenever this makes sense,*

$$\beta_{ij;k} := \arccos\left(\frac{\cos(\alpha_{ij}) + \cos(\alpha_{ik})\cos(\alpha_{jk})}{\sin(\alpha_{ik})\sin(\alpha_{jk})}\right).$$

(*If conditions* (ii) *and* (iii) *below are satisfied, then* $\beta_{ij;k}$ *is well-defined.*)

Then $\alpha \in P_4$ (*defined in Proposition* 1) *if and only if, for all permutations* i, j, k, l *of* 1, 2, 3, 4, *the following conditions are satisfied*:

(i) $0 < \alpha_{ij} < \pi$;

(ii) $\alpha_{ij} + \alpha_{ik} + \alpha_{jk} \geq \pi$;

(iii) $-\alpha_{ij} + \alpha_{ik} + \alpha_{jk} < \pi$;

(iv) $\beta_{ij;l} + \beta_{ik;l} + \beta_{jk;l} < \pi$.

Proof. Let $A := (a_{ij})^4_{i,j=1} := \tilde{A}(\alpha)$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(a) We first verify that conditions (i)–(iv) guarantee that $\alpha \in P_4$. The equations

$$\begin{aligned} A_{ll}^{\mathrm{ad}} &= -1 + a_{ij}^2 + a_{ik}^2 + a_{jk}^2 + 2a_{ij}a_{ik}a_{jk} \\ &= (a_{ij} + a_{ik}a_{jk})^2 - (1 - a_{ik}^2)(1 - a_{jk}^2) \\ &= [\cos(\alpha_{ij}) + \cos(\alpha_{ik})\cos(\alpha_{jk})]^2 - \sin^2(\alpha_{ik})\sin^2(\alpha_{jk}) \\ &= [\cos(\alpha_{ij}) + \cos(\alpha_{ik} + \alpha_{jk})][\cos(\alpha_{ij}) + \cos(\alpha_{ik} - \alpha_{jk})] \end{aligned}$$

yield the decomposition

$$(6) \qquad \begin{cases} A_{ll}^{\mathrm{ad}} = 4\cos\left(\dfrac{\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right)\cos\left(\dfrac{-\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right) \\ \qquad \times \cos\left(\dfrac{\alpha_{ij} - \alpha_{ik} + \alpha_{jk}}{2}\right)\cos\left(\dfrac{\alpha_{ij} + \alpha_{ik} - \alpha_{jk}}{2}\right). \end{cases}$$

Conditions (ii) and (iii) therefore furnish $A_{ll}^{\mathrm{ad}} \leq 0$. Furthermore, they imply

$$\begin{aligned} 0 &\geq 2\cos\left(\frac{\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right)\cos\left(\frac{-\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right) \\ &= \cos(\alpha_{ij}) + \cos(\alpha_{ik} + \alpha_{jk}) \implies \frac{\cos(\alpha_{ij}) + \cos(\alpha_{ik})\cos(\alpha_{jk})}{\sin(\alpha_{ik})\sin(\alpha_{jk})} \leq 1 \end{aligned}$$

and

$$\begin{aligned} 0 &\leq 2\cos\left(\frac{\alpha_{ij} - \alpha_{ik} + \alpha_{jk}}{2}\right)\cos\left(\frac{\alpha_{ij} + \alpha_{ik} - \alpha_{jk}}{2}\right) \\ &= \cos(\alpha_{ij}) + \cos(\alpha_{ik} - \alpha_{jk}) \implies \frac{\cos(\alpha_{ij}) + \cos(\alpha_{ik})\cos(\alpha_{jk})}{\sin(\alpha_{ik})\sin(\alpha_{jk})} \geq -1; \end{aligned}$$

hence the numbers $\beta_{ij;k}$ are well-defined. Gauß' algorithm and equation (6) then yield

$$(7) \qquad \begin{cases} \det A = -\sin^2(\alpha_{il})\sin^2(\alpha_{jl})\sin^2(\alpha_{kl})\det\tilde{A}(\beta_{ij;l}, \beta_{ik;l}, \beta_{jk;l}) \\ \quad = -4\sin^2(\alpha_{il})\sin^2(\alpha_{jl})\sin^2(\alpha_{kl})\cos\left(\dfrac{\beta_{ij;l} + \beta_{ik;l} + \beta_{jk;l}}{2}\right) \\ \qquad \times \cos\left(\dfrac{-\beta_{ij;l} + \beta_{ik;l} + \beta_{jk;l}}{2}\right)\cos\left(\dfrac{\beta_{ij;l} - \beta_{ik;l} + \beta_{jk;l}}{2}\right) \\ \qquad \times \cos\left(\dfrac{\beta_{ij;l} + \beta_{ik;l} - \beta_{jk;l}}{2}\right) \end{cases}$$

and condition (iv) implies that $\det A < 0$. Hence the number of sign changes in the sequence

$$1, -1, \sin^2(\alpha_{ij}), A_{ll}^{\mathrm{ad}}, \det A$$

is exactly three and Jacobi's theorem shows that condition (i) in the definition of P_n is satisfied. Finally, with $s := \sin(\alpha_{il})\sin(\alpha_{jl})\sin^2(\alpha_{kl})$, a straightforward calculation gives

$$
(8)\qquad \begin{cases} A_{ij}^{\mathrm{ad}} = -s[\cos(\beta_{ij;l}) + \cos(\beta_{ik;l})\cos(\beta_{jk;l})] \\ \qquad \leq -s[\cos(\beta_{ij;l}) + \cos(\beta_{ik;l} + \beta_{jk;l})] \\ \qquad = -2s\cos\left(\dfrac{\beta_{ij;l} + \beta_{ik;l} + \beta_{jk;l}}{2}\right)\cos\left(\dfrac{-\beta_{ij;l} + \beta_{ik;l} + \beta_{jk;l}}{2}\right) < 0 \end{cases}
$$

due to (iv). Hence both conditions in the definition of P_n hold and $\alpha \in P_4$.

(b) Conversely, if $\alpha \in P_4$, then $A_{ll}^{\mathrm{ad}} \leq 0$. Since condition (i) is fulfilled due to (4) (see also the end of the proof of Proposition 1), two of the factors in (6) are always positive (e.g., the last two if $\alpha_{ij} \leq \alpha_{ik}$ and $\alpha_{ij} \leq \alpha_{jk}$). Hence we conclude that

$$
\cos\left(\frac{\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right) \leq 0, \qquad \cos\left(\frac{-\alpha_{ij} + \alpha_{ik} + \alpha_{jk}}{2}\right) \geq 0
$$

and thus

$$
\alpha_{ij} + \alpha_{ik} + \alpha_{jk} \geq \pi, \qquad -\alpha_{ij} + \alpha_{ik} + \alpha_{jk} \leq \pi.
$$

Therefore, the set of angles β is well-defined. If we had $-\alpha_{ij} + \alpha_{ik} + \alpha_{jk} = \pi$, then we would obtain $\beta_{jk;i} = \pi$ and (8) would imply $A_{jk}^{\mathrm{ad}} \geq 0$ in contradiction to condition (ii) in the definition of P_n. Therefore, conditions (ii), (iii) in the proposition must hold. Similarly as in (8), we obtain

$$
A_{ij}^{\mathrm{ad}} \geq -2s\cos\left(\frac{\beta_{ij;l} + \beta_{ik;l} - \beta_{jk;l}}{2}\right)\cos\left(\frac{\beta_{ij;l} - \beta_{ik;l} + \beta_{jk;l}}{2}\right),
$$

and hence $A_{ij}^{\mathrm{ad}} < 0$ implies $\beta_{ij;l} + \beta_{ik;l} - \beta_{jk;l} < \pi$. But then three of the cosine factors in (7) are positive and from $\det A < 0$ we infer condition (iv). □

Remark. In particular, Proposition 2 shows that $\mathring{P}_4$ is the set of all $\alpha \in P_4$ such that

$$
\forall \text{ pairwise different } i,j,k \in \{1,2,3,4\} : \alpha_{ij} + \alpha_{ik} + \alpha_{jk} > \pi.
$$

More precisely, equation (6) combined with condition (iii) in Proposition 2 yield that, for $C = h_4(\alpha, d) \in Q_4$ and $\{i,j,k,l\} = \{1,2,3,4\}$, $c_{ll} = 0$ if and only if $\alpha_{ij} + \alpha_{ik} + \alpha_{jk} = \pi$.

As follows from the next lemma, which we shall need in Section 3, $\mathring{P}_4$ is connected and hence yields one of the components of M_4 (comp. the observation before Proposition 1).

Lemma. *All the sets* $P_4, \mathring{P}_4, P_4 \setminus \mathring{P}_4, \{\alpha \in P_4 : \alpha_{12} + \alpha_{13} + \alpha_{23} = \pi\}$, *and*

$$
\{\alpha \in P_4 : \alpha_{12} + \alpha_{13} + \alpha_{23} = \pi, \forall 1 \leq i < j \leq 3 : \alpha_{ij} + \alpha_{i4} + \alpha_{j4} > \pi\}
$$

are arcwise connected.

Proof. We start with observing that

$$
(9)\qquad \begin{cases} P_4' := \{\alpha \in P_4 : \forall \text{ pairwise different } i,j,k \in \{1,2,3,4\} \\ \qquad\qquad : \alpha_{ij} + \alpha_{ik} + \alpha_{jk} = \pi\} \end{cases}
$$

is arcwise connected. In fact, for $\alpha \in P_4'$, we have $\alpha_{12} = \alpha_{34}$, $\alpha_{13} = \alpha_{24}$ and

$\alpha_{23} = \alpha_{14}$, and, on the other hand, each $\alpha \in]0,\pi[^6$ fulfilling these relations and $\alpha_{12} + \alpha_{13} + \alpha_{23} = \pi$ belongs to P_4', since then all the numbers $\beta_{ij;k}$ vanish. Hence

$$P_4' = \{(u, v, \pi - u - v, \pi - u - v, v, u) : u > 0, v > 0, u + v < \pi\}$$

clearly is arcwise connected.

We show next that each $\alpha \in P_4$ can be joined by a path in P_4 to some $\alpha' \in P_4'$ thus completing the proof that P_4 is arcwise connected. If $C := (c_{ij})_{i,j=1}^4 := \tilde{A}(\alpha)^{-1}$ and

$$C(t) := (c_{ij}(t))_{i,j=1}^4, \quad c_{ij}(t) := \begin{cases} tc_{11} : & i = j = 1, \\ c_{ij} : & \text{else,} \end{cases}$$

then $C(t)$, $0 \le t \le 1$, is a curve in Q_4. In fact, $C(1) = C \in Q_4$ and, by (4),

$$\frac{\partial \det C(t)}{\partial t} = c_{11} C(t)_{11}^{\mathrm{ad}} \ge 0 \quad \text{if } C(t) \in Q_4.$$

This implies $\det C(t) < 0$ and $C(t) \in Q_4$ for $0 \le t \le 1$. (Note that the number of positive eigenvalues cannot change on the continuous path $C(t)$.) We therefore can join C in Q_4 with a matrix with vanishing diagonal and hence can join α in P_4 with $\alpha' \in P_4'$.

Let us finally consider the set $\overset{\circ}{P}_4$, the argument being similar for the remaining cases. If α and α' are in $\overset{\circ}{P}_4$ and $\alpha(t)$, $0 \le t \le 1$, is a path in P_4 connecting them, then, for a suitable small $\epsilon > 0$, $\alpha(t) + t(1-t)\epsilon(1,1,1,1,1,1)$ is a path in $\overset{\circ}{P}_4$ connecting α and α'. □

3. VOLUME FORMULA FOR ASYMPTOTIC HYPERBOLIC TETRAHEDRA

We first state the announced representation of

$$f_4(C) = \sqrt{|\det C|} \int_{\mathbb{R}_+^4} \mathrm{e}^{-\langle Cx, x\rangle} \, \mathrm{d}x$$

by 15 Cl_2-functions containing as arguments linear combinations of α_{ij}, if $C = h_4(\alpha, d) \in Q_4 \backslash (Q_4 \cap N_4)$. (For simplicity, we shall assume $c_{44} = 0$.) Thereafter it will be shown that $f_4(C)$ equals half the volume of a tetrahedron in hyperbolic space and with dihedral angles α_{ij}. The conditions $c_{44} = 0$ or, equivalently, $\alpha_{12} + \alpha_{13} + \alpha_{23} = \pi$ mean that this tetrahedron is asymptotic, i.e., that one of its vertices lies at infinity.

Let us recall the definition of Clausen's function, the version of dilogarithm we need here (cf. [11, (4.4), (4.5), p. 102]):

$$\begin{aligned} \mathrm{Cl}_2(x) &:= -\int_0^x \ln\left(2\left|\sin\frac{\theta}{2}\right|\right) \mathrm{d}\theta \\ &= \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}, \quad x \in \mathbb{R}. \end{aligned}$$

The first equation shows that Cl_2 is real analytic for all real x which are not multiples of 2π, the second one implies that Cl_2 is periodic with period 2π.

Theorem 2. *Let Q_n and h_n be defined as in Proposition* 1, $C := (c_{ij})_{i,j=1}^4 := h_4(\alpha, d) \in Q_4$ *and suppose that* $c_{44} = 0$, *i.e.*, $\alpha_{12} + \alpha_{13} + \alpha_{23} = \pi$. *Then*

$$
(10)\quad \begin{cases} \sqrt{|\det C|} \int\limits_{\mathbb{R}_+^4} \mathrm{e}^{-\langle Cx, x\rangle} \,\mathrm{d}x = \frac{1}{8} [\mathrm{Cl}_2(2\alpha_{12}) + \mathrm{Cl}_2(2\alpha_{13}) + \mathrm{Cl}_2(2\alpha_{23}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{12} + \alpha_{14} + \alpha_{24}) - \mathrm{Cl}_2(\pi + \alpha_{12} + \alpha_{14} - \alpha_{24}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{12} - \alpha_{14} + \alpha_{24}) - \mathrm{Cl}_2(\pi + \alpha_{12} - \alpha_{14} - \alpha_{24}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{13} + \alpha_{14} + \alpha_{34}) - \mathrm{Cl}_2(\pi + \alpha_{13} + \alpha_{14} - \alpha_{34}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{13} - \alpha_{14} + \alpha_{34}) - \mathrm{Cl}_2(\pi + \alpha_{13} - \alpha_{14} - \alpha_{34}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{23} + \alpha_{24} + \alpha_{34}) - \mathrm{Cl}_2(\pi + \alpha_{23} + \alpha_{24} - \alpha_{34}) \\ \quad - \mathrm{Cl}_2(\pi + \alpha_{23} - \alpha_{24} + \alpha_{34}) - \mathrm{Cl}_2(\pi + \alpha_{23} - \alpha_{24} - \alpha_{34})]. \end{cases}
$$

Proof. (a) Without loss of generality, $d_1 = \cdots = d_4 = 1$ can be assumed from the outset (comp. Remark 1 to Proposition 1). For $\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{24}, \alpha_{34}) \in \mathbb{R}^5$, set

$$\hat{\alpha} := (\alpha_{12}, \alpha_{13}, \alpha_{14}, \pi - \alpha_{12} - \alpha_{13}, \alpha_{24}, \alpha_{34})$$

and define the mapping

$$F : R := \{\alpha \in]0, \pi[^5 : \hat{\alpha} \in P_4\} \longrightarrow \mathbb{R} :$$

$$\alpha \longmapsto \frac{1}{\sqrt{|\det \tilde{A}(\hat{\alpha})|}} \int\limits_{\mathbb{R}_+^4} \mathrm{e}^{-\langle \tilde{A}(\hat{\alpha})^{-1}x, x\rangle} \,\mathrm{d}x.$$

Let us first prove that F is continuous. If $C \in Q_4$, the introduction of polar coordinates $x = r\omega$ yields

$$\int\limits_{\mathbb{R}_+^4} \mathrm{e}^{-\langle Cx, x\rangle} \,\mathrm{d}x = \frac{1}{2} \int\limits_{\mathbb{R}_+^4 \cap \mathbb{S}_3} \frac{\mathrm{d}\sigma(\omega)}{\langle C\omega, \omega\rangle^2}.$$

The denominator of the fraction in the last integral can vanish on the axes only (cf. (b) of Proposition 1). Since, e.g., at the point $(0, 0, 0, 1)$ and in the local coordinates $\omega' = (\omega_1, \omega_2, \omega_3)$, the estimate $\langle C\omega, \omega\rangle \geq \epsilon|\omega'|$ holds for some $\epsilon > 0$, small $\omega' \in \mathbb{R}_+^3$, and uniformly with respect to C in compact subsets of Q_4, Lebesgue's dominated convergence theorem implies the continuity of f_4 on Q_4 and hence of F.

(b) We next show that F is continuously differentiable on

$$\overset{\circ}{R} = \{\alpha \in]0, \pi[^5 : \hat{\alpha} \in P_4, \alpha_{12} + \alpha_{14} + \alpha_{24} > \pi, \alpha_{13} + \alpha_{14} + \alpha_{34} > \pi, \\ \alpha_{24} + \alpha_{34} > \alpha_{12} + \alpha_{13}\}.$$

For this, we have to verify that, e.g., the functions

$$\left| \frac{\partial}{\partial \alpha_{12}} \mathrm{e}^{-\langle Cx, x\rangle} \right|, \quad C := (c_{ij})_{i,j=1}^4 := \tilde{A}(\hat{\alpha})^{-1},$$

can be estimated by a fixed integrable function on $\mathbb{R}_+^4$ if α stays in a compact subset K of $\overset{\circ}{R}$. The calculation in the proof of Theorem 1 furnishes

$$\frac{\partial}{\partial\alpha_{12}} \mathrm{e}^{-\langle Cx,x\rangle} = -2 \sum_{k=1}^{4} \sum_{l=1}^{4} (c_{1k}\sin(\alpha_{12}) - c_{3k}\sin(\hat{\alpha}_{23}))c_{2l}x_k x_l \,\mathrm{e}^{-\langle Cx,x\rangle}.$$

For $\alpha \in R$, we have

$$c_{14}^2 = c_{14}^2 - c_{11}c_{44} = -\sin^2(\hat{\alpha}_{23})\det C, \qquad c_{34}^2 = -\sin^2(\alpha_{12})\det C,$$

and hence $c_{14}\sin(\alpha_{12}) - c_{34}\sin(\hat{\alpha}_{23}) = 0$. Therefore, there exists $\epsilon > 0$ such that for all $\alpha \in K$ and $x = (x', x_4) \in \mathbb{R}_+^4$ the estimates

$$\langle Cx, x\rangle \geq \epsilon(|x'|^2 + |x'|x_4),$$

$$\left|\frac{\partial}{\partial\alpha_{12}} \mathrm{e}^{-\langle Cx,x\rangle}\right| \leq \frac{1}{\epsilon}|x'|(|x'| + x_4)\,\mathrm{e}^{-\epsilon|x'|^2}\,\mathrm{e}^{-\epsilon|x'|x_4}$$

hold. Since the last function is in $L^1(\mathbb{R}_+^4)$, and since a similar argument applies for the derivatives $\partial/\partial\alpha_{i4}$, $i = 1,2,3$, it results that F is continuously differentiable on $\mathring{R}$.

(c) Now Schläfli's formula (3) can be applied in order to calculate $\mathrm{d}F$ on $\mathring{R}$. As above, let $A := (a_{ij})_{i,j=1}^4 := \tilde{A}(\hat{\alpha})$ and $C := (c_{ij})_{i,j=1}^4 := A^{-1}$. Recall that $C_{[14]}$ denotes the submatrix of C produced by deleting the first and fourth rows and columns. It follows from (4) that $\det C_{[14]} < 0$ and hence (cf. [16, Proposition 5, p. 101])

$$\sqrt{|\det C_{[14]}|} \int_{\mathbb{R}_+^2} \mathrm{e}^{-\langle C_{[14]}x,x\rangle}\,\mathrm{d}x = \frac{1}{4}\ln\left(\frac{c_{23} + \sqrt{c_{23}^2 - c_{22}c_{33}}}{c_{23} - \sqrt{c_{23}^2 - c_{22}c_{33}}}\right).$$

A repetition of the reasoning in the proof of Theorem 1 then yields

$$\begin{aligned}\frac{\partial F}{\partial\alpha_{14}} &= \frac{1}{8}\ln\left(\frac{c_{23} - \sqrt{c_{23}^2 - c_{22}c_{33}}}{c_{23} + \sqrt{c_{23}^2 - c_{22}c_{33}}}\right)\\ &= \frac{1}{8}\ln\left(\frac{-A_{23}^{\mathrm{ad}} - \sin(\alpha_{14})\sqrt{|\det A|}}{-A_{23}^{\mathrm{ad}} + \sin(\alpha_{14})\sqrt{|\det A|}}\right)\\ &= \frac{1}{8}\ln\left(\frac{-A_{23}^{\mathrm{ad}} + \sin(\alpha_{14})A_{14}^{\mathrm{ad}}/\sin(\hat{\alpha}_{23})}{-A_{23}^{\mathrm{ad}} - \sin(\alpha_{14})A_{14}^{\mathrm{ad}}/\sin(\hat{\alpha}_{23})}\right).\end{aligned}$$

Here the identities

$$c_{23} = -\frac{A_{23}^{\mathrm{ad}}}{|\det A|}, \qquad c_{23}^2 - c_{22}c_{33} = \frac{a_{14}^2 - a_{11}a_{44}}{\det A} = \frac{\sin^2(\alpha_{14})}{|\det A|}$$

and

$$(A_{14}^{\mathrm{ad}})^2 = (A_{14}^{\mathrm{ad}})^2 - A_{11}^{\mathrm{ad}}A_{44}^{\mathrm{ad}} = \sin^2(\hat{\alpha}_{23})\,|\det A|$$

were used. In order to obtain $\partial F/\partial\alpha_{12}$ and $\partial F/\partial\alpha_{13}$, we can apply a limit argument with respect to c_{44} for $c_{44} \searrow 0$. (In fact, since the function $t\,\mathrm{e}^{-t}$ is bounded on $\mathbb{R}_+$, the same dominating function as in (b) can be used.) Schläfli's formula then yields

$$\frac{\partial F}{\partial \alpha_{12}} = \lim_{c_{44}\searrow 0} \frac{1}{8} \left[\ln\left(\frac{c_{34} - \sqrt{c_{34}^2 - c_{33}c_{44}}}{c_{34} + \sqrt{c_{34}^2 - c_{33}c_{44}}} \right) - \ln\left(\frac{c_{14} - \sqrt{c_{14}^2 - c_{11}c_{44}}}{c_{14} + \sqrt{c_{14}^2 - c_{11}c_{44}}} \right) \right]$$

$$= \lim_{c_{44}\searrow 0} \frac{1}{8} \ln\left(\frac{1 - \sqrt{1 - c_{33}c_{44}/c_{34}^2}}{1 - \sqrt{1 - c_{11}c_{44}/c_{14}^2}} \right)$$

$$= \frac{1}{8} \ln\left(\frac{c_{14}^2 c_{33}}{c_{34}^2 c_{11}} \right) = \frac{1}{8} \ln\left(\frac{\sin^2(\hat{\alpha}_{23}) A_{33}^{\text{ad}}}{\sin^2(\alpha_{12}) A_{11}^{\text{ad}}} \right).$$

(d) In the next step, we verify that the differentials of the left-hand and of the right-hand sides in (10) coincide in $\overset{\circ}{R}$. For the derivatives with respect to α_{12} and α_{13}, respectively, equation (6) immediately yields the result. What considers $\partial/\partial\alpha_{14}$, say, a straightforward calculation gives

$$-A_{23}^{\text{ad}} \pm \frac{\sin(\alpha_{14})}{\sin(\hat{\alpha}_{23})} A_{14}^{\text{ad}} = [\cos(\alpha_{24}) + \cos(\alpha_{12} \pm \alpha_{14})][\cos(\alpha_{34}) + \cos(\alpha_{13} \pm \alpha_{14})]$$

$$= 4\cos\left(\frac{\alpha_{12} \pm \alpha_{14} + \alpha_{24}}{2} \right) \cos\left(\frac{\alpha_{12} \pm \alpha_{14} - \alpha_{24}}{2} \right)$$

$$\times \cos\left(\frac{\alpha_{13} \pm \alpha_{14} + \alpha_{34}}{2} \right) \cos\left(\frac{\alpha_{13} \pm \alpha_{14} - \alpha_{34}}{2} \right),$$

and hence also $\partial F/\partial\alpha_{14}$ coincides with the derivative with respect to α_{14} of the right-hand side of (10).

(e) Finally, the fact that R and $\overset{\circ}{R}$ are connected due to the lemma is employed to conclude that the two sides of (10) can, according to (d), differ on $\overset{\circ}{R}$ by a constant at most. The continuity of F shown in (a) implies that the same holds true on R. In order to prove that this constant actually vanishes, let us consider the limit for $\epsilon \searrow 0$ when $\alpha(\epsilon) := (\epsilon, \pi/2, \pi/2 - \epsilon, \pi/2, \epsilon) \in R$, i.e.,

$$\tilde{A}(\widehat{\alpha(\epsilon)}) = \begin{pmatrix} -1 & \cos\epsilon & 0 & \sin\epsilon \\ \cos\epsilon & -1 & \sin\epsilon & 0 \\ 0 & \sin\epsilon & -1 & \cos\epsilon \\ \sin\epsilon & 0 & \cos\epsilon & -1 \end{pmatrix}$$

and

$$C(\epsilon) := \tilde{A}(\widehat{\alpha(\epsilon)})^{-1} = \frac{1}{\sin(2\epsilon)} \begin{pmatrix} 0 & \sin\epsilon & 1 & \cos\epsilon \\ \sin\epsilon & 0 & \cos\epsilon & 1 \\ 1 & \cos\epsilon & 0 & \sin\epsilon \\ \cos\epsilon & 1 & \sin\epsilon & 0 \end{pmatrix}, \quad \sqrt{|\det C(\epsilon)|} = \frac{1}{\sin(2\epsilon)}.$$

(Note that $\widehat{\alpha(\epsilon)}$ belongs to the set P_4' defined in (9) and hence all $\beta_{ij;k}$ vanish. We then obtain $C(\epsilon)$ easily from formulae (7) and (8).) Now, on the one hand, the limit

$$\lim_{\epsilon\searrow 0} \int_{\mathbb{R}_+^4} \frac{1}{\epsilon} \exp(-x_1x_2 - x_3x_4 - (x_1 + x_2)(x_3 + x_4)/\epsilon)\, \mathrm{d}x$$

vanishes due to the dominated convergence theorem (since the integrated functions are dominated by some multiple of

$$\frac{\exp(-x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4)}{(x_1+x_2)(x_3+x_4)} \in L^1(\mathbb{R}_+^4)),$$

and, on the other hand, the right-hand side in (10), evaluated for $\widehat{\alpha(\epsilon)}$, obviously tends to zero if $\epsilon \searrow 0$. This completes the proof. $\square$

In order to interpret (10) as a volume formula for asymptotic tetrahedra in hyperbolic space, let us recall some facts from hyperbolic geometry (see [2], [17] for more details).

If $\mathbb{R}^n$ is equipped with the Lorentz scalar product

$$[x,y] := -x_1y_1 - \cdots - x_{n-1,n-1}y_{n-1,n-1} + x_ny_n, \quad x,y \in \mathbb{R}^n,$$

and with the metric $(\sum_{i=1}^{n-1} \mathrm{d}x_i \otimes \mathrm{d}x_i) - \mathrm{d}x_n \otimes \mathrm{d}x_n$, then the *hyperboloid model* of hyperbolic space is given by

$$\mathbb{H}_{n-1} := \{x \in \mathbb{R}^n : [x,x] = 1, x_n > 0\}$$

endowed with the induced metric. This is a Riemannian metric on $\mathbb{H}_{n-1}$, and we denote it by G. The corresponding distance $d(x,y)$ of $x, y \in \mathbb{H}_{n-1}$ is given by $d(x,y) = \operatorname{arcosh}([x,y])$. In the co-ordinates $x' = (x_1, \ldots, x_{n-1})$, G is represented by

$$G = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} g_{ij}\,\mathrm{d}x_i \otimes \mathrm{d}x_j = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\delta_{ij} - \frac{x_ix_j}{x_n^2}\right) \mathrm{d}x_i \otimes \mathrm{d}x_j.$$

(δ_{ij} denotes the Kronecker symbol.) The only non-unity eigenvalue of the matrix $(g_{ij})_{i,j=1}^{n-1}$ is the eigenvalue x_n^{-2} corresponding to eigenvectors parallel with x'. Hence $\det((g_{ij})_{i,j=1}^{n-1}) = x_n^{-2}$, and the *canonical volume measure* induced by G is $\mathrm{d}\mu := x_n^{-1}\,\mathrm{d}x'$. Here $\mathrm{d}x'$ denotes Lebesgue measure in the co-ordinates $x_1, \ldots, x_{n-1}$, i.e. the positive measure associated with the $(n-1)$-form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n-1}$ on $\mathbb{H}_{n-1}$.

If $V := (v^{(1)}, \ldots, v^{(n)})$ is an n-tuple of linearly independent vectors in $\mathbb{R}^n$ satisfying $[v^{(i)}, v^{(i)}] \geq 0$ and $v_n^{(i)} > 0$, $i = 1, \ldots, n$, then we denote by $C(V)$ the matrix

$$C(V) := (c_{ij})_{i,j=1}^n, \quad c_{ij} := [v^{(i)}, v^{(j)}],$$

and by $S(V)$ the following $(n-1)$-dimensional *simplex*:

$$S(V) := \left\{v \in \mathbb{H}_{n-1} : \exists\lambda_1 \geq 0, \ldots, \exists\lambda_n \geq 0 : v = \sum_{i=1}^n \lambda_i v^{(i)}\right\}.$$

For $n = 3$ or $n = 4$, $S(V)$ is called a *triangle* or a *tetrahedron*, respectively.

$\mathbb{H}_{n-1}$ can be embedded into its *closure* $\overline{\mathbb{H}_{n-1}} := \mathbb{H}_{n-1} \cup \partial\mathbb{H}_{n-1}$, where $\partial\mathbb{H}_{n-1}$, the *boundary* of $\mathbb{H}_{n-1}$, is, e.g., defined by

$$\partial\mathbb{H}_{n-1} := \{\{\lambda v : \lambda > 0\} : v \in \mathbb{R}^n, [v,v] = 0, v_n > 0\}.$$

$\overline{\mathbb{H}_{n-1}}$ is topologized in an obvious way and then is homeomorphic to a closed ball in $\mathbb{R}^{n-1}$ (see [2, A.5]). Points in $\partial\mathbb{H}_{n-1}$ are called points *at infinity*. With this definition, the closure of the simplex $S(V)$ has exactly n vertices $x^{(i)} \in \overline{\mathbb{H}_{n-1}}$, $i = 1, \ldots, n$, namely

$$x^{(i)} = \begin{cases} v^{(i)}/\sqrt{[v^{(i)}, v^{(i)}]} & : [v^{(i)}, v^{(i)}] > 0, \\ \{\lambda v^{(i)} : \lambda > 0\} & : \text{else.} \end{cases}$$

The simplex $S(V)$ is called *asymptotic* if at least one of its vertices lies at infinity.

Let us assume now that $n > 2$ and show that $C(V) \in Q_n$ and that, in this way, each matrix $C \in Q_n$ is obtained. In fact, the matrix $C(V)$ is congruent to the coefficient matrix of the Lorentz metric $[x, y]$ and hence, by Sylvester's law of inertia, condition (i) in the definition of Q_n (see Proposition 1) holds. The second condition, i.e., $c_{ii} \geq 0$ and $c_{ij} > 0$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$, is also true since the vectors $v^{(i)}$ all belong to the cone $\{x \in \mathbb{R}^n : [x, x] \geq 0, x_n > 0\}$. Conversely, if $C = (c_{ij})_{i,j=1}^n \in Q_n$, and if the quadratic form $\langle Cx, x\rangle$ is written as a sum of squares, i.e., $\langle Cx, x\rangle = [Vx, Vx]$, $x \in \mathbb{R}^n$, for some $V \in \mathrm{gl}_n$, then the columns of V can be taken as $v^{(i)}$, $i = 1, \ldots, n$, provided we choose V or $-V$ according to whether one and hence all of the numbers $v_n^{(i)}$ are positive or negative, respectively. (This relationship between matrices in Q_n and simplices in $\mathbb{H}_{n-1}$ yields a geometric proof of the connectedness assertions in the lemma above. In fact, each $(n-1)$-dimensional simplex in $\mathbb{H}_{n-1}$ can continuously be deformed into each other.)

Let us next define $\alpha(V) \in P_n$ by the equation $h_n(\alpha(V), d) = C(V)$ for some $d \in]0, \infty[^n$ (cf. Proposition 1). If $\alpha(V) = (\alpha_{12}, \ldots, \alpha_{1n}, \alpha_{23}, \ldots, \alpha_{n-1,n})$, then α_{ij}, $1 \leq i < j \leq n$, are the *dihedral angles* in the simplex $S(V)$, i.e., α_{ij} is the angle between the faces ϵ_1, ϵ_2 of $S(V)$ spanned by $x^{(1)}, \ldots, x^{(i-1)}, x^{(i+1)}, \ldots, x^{(n)}$ and $x^{(1)}, \ldots, x^{(j-1)}, x^{(j+1)}, \ldots, x^{(n)}$, respectively. In fact, let $y, z \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$

$$[x, y] = \det(x, v^{(j)}, v^{(1)}, \ldots, v^{(i-1)}, v^{(i+1)}, \ldots, v^{(j-1)}, v^{(j+1)}, \ldots, v^{(n)}),$$

$$[x, z] = \det(x, v^{(i)}, v^{(1)}, \ldots, v^{(i-1)}, v^{(i+1)}, \ldots, v^{(j-1)}, v^{(j+1)}, \ldots, v^{(n)}).$$

Then y and z belong to the tangent space at any common point in $\mathbb{H}_{n-1}$ of the faces ϵ_1, ϵ_2, and they stand normal on ϵ_1 and on ϵ_2, respectively. By the definition of the metric on $\mathbb{H}_{n-1}$, we obtain, for the dihedral angle γ_{ij}, the formula

$$\gamma_{ij} = \arccos\left(\frac{-[y, z]}{\sqrt{[y, y][z, z]}}\right).$$

But since, for some constant u,

$$y = u \sum_{k=1}^{n} (C^{-1})_{ik} v^{(k)}, \qquad z = -u \sum_{k=1}^{n} (C^{-1})_{jk} v^{(k)},$$

we conclude that

$$\gamma_{ij} = \arccos\left(\frac{(C^{-1})_{ij}}{\sqrt{(C^{-1})_{ii}(C^{-1})_{jj}}}\right) = \alpha_{ij}.$$

The next theorem is enunciated for abstract three-dimensional hyperbolic space denoted by X and without referring to a special model. That simply means that X is a Riemannian space isometric to $\mathbb{H}_3$.

Theorem 3. *Let X denote three-dimensional hyperbolic space and let*

$$\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) \in \mathbb{R}^6.$$

Then we have:

(a) *α represents the dihedral angles of a tetrahedron in X (asymptotic or not) if and only if $\alpha \in P_4$ or, equivalently, if and only if conditions* (i)–(iv) *in Proposition* 2 *are satisfied.*

(b) *If α represents the dihedral angles of an asymptotic tetrahedron S in X the fourth vertex of which tetrahedron lies at infinity, i.e., $\alpha_{12} + \alpha_{13} + \alpha_{23} = \pi$, then the volume of S equals twice the right-hand side of formula* (10) *in Theorem* 2.

Proof. (a) We have seen above that, for $S = S(V)$, the matrix $C(V)$ belongs to P_4 and that all matrices in P_4 occur in this way. Hence statement (a) is a consequence of Proposition 2.

(b) We first consider general $n > 2$. Let $V := (v^{(1)}, \dots, v^{(n)})$ be as above and set $C := C(V)$. Then

$$\forall x \in \mathbb{R}^n : \langle Cx, x\rangle = [Vx, Vx]$$

if, by abuse of notation, V also denotes the matrix in gl_n with columns $v^{(i)}$, $i = 1, \dots, n$. Since $|\det V| = \sqrt{|\det C|}$, the substitution $y = Vx$ yields

$$\sqrt{|\det C|}\int_{\mathbb{R}^n_+} e^{-\langle Cx, x\rangle}\, dx = \int_{V\mathbb{R}^n_+} e^{-[y,y]}\, dy. \tag{11}$$

Next, polar co-ordinates are introduced. The mapping

$$]0, \infty[\ \times \mathbb{H}_{n-1} \longrightarrow \{x \in \mathbb{R}^n : [x, x] > 0, x_n > 0\} : (r, \omega) \longmapsto r\omega$$

is a diffeomorphism and the pull-back of the n-form $dx_1 \wedge \cdots \wedge dx_n$ is, by a straightforward calculation, $r^{n-1}\, dr \wedge L$, where

$$L := \sum_{i=1}^{n} (-1)^{i-1}\omega_i\, d\omega_1 \wedge \cdots \wedge d\omega_{i-1} \wedge d\omega_{i+1} \wedge \cdots \wedge d\omega_n$$

is the Leray form on $\mathbb{H}_{n-1}$. Because of $\omega_n\, d\omega_n = \sum_{i=1}^{n-1} \omega_i\, d\omega_i$, we obtain

$$\begin{aligned} L &= (-1)^n\left[-\omega_n + \sum_{i=1}^{n-1} \frac{\omega_i^2}{\omega_n}\right] d\omega_1 \wedge \cdots \wedge d\omega_{n-1} \\ &= \frac{(-1)^{n-1}}{\omega_n}\, d\omega_1 \wedge \cdots \wedge d\omega_{n-1}. \end{aligned}$$

For the associated positive measures, this yields $\mathrm{d}x = r^{n-1}\,\mathrm{d}r \otimes \mathrm{d}\mu$, where $\mathrm{d}x$ and $\mathrm{d}r$ stand here for the Lebesgue measures on $\{x \in \mathbb{R}^n : [x,x] > 0, x_n > 0\}$ and on $]0,\infty[$, respectively. Hence

$$(12) \qquad \int_{V\mathbb{R}^n_+} \mathrm{e}^{-[y,y]}\,\mathrm{d}y = \int_0^\infty r^{n-1}\mathrm{e}^{-r^2}\,\mathrm{d}r \int_{V\mathbb{R}^n_+ \cap \mathbb{H}_{n-1}} \mathrm{d}\mu = \frac{1}{2}\,\Gamma\left(\frac{n}{2}\right)|S(V)|$$

if $|S(V)|$ denotes the volume of $S(V)$. If we now specialize to the case of an asymptotic tetrahedron the fourth vertex $x^{(4)}$ of which lies at infinity, equations (11), (12) combined with Theorem 2 imply assertion (b). □

Remarks. 1. The statement in (a) answers the problem posed in [6, VII.6, p. 174] to find conditions which characterize the set of dihedral angles of non-asymptotic tetrahedra in hyperbolic space. In this book, the necessary conditions

(i) $\forall 1 \le i < j \le 4 : \alpha_{ij} \in]0,\pi[$,

(ii) $\forall 1 \le i < j < k \le 4 : \alpha_{ij} + \alpha_{ik} + \alpha_{jk} > \pi$ (comp. the remark to Proposition 2),

(iii) $\det \tilde{A}(\alpha) < 0$,

are stated and the question on the missing additional condition is put forward. Since (i)–(iii) above imply that the matrix $\tilde{A}(\alpha)$ has one positive and three negative eigenvalues, what is still missing is condition (ii) in the definition of P_n (see Proposition 1). However, (i)–(iv) in Proposition 2 seem to constitute a more explicit set of conditions. We observe that the angle $\beta_{ij;k}$ introduced in Proposition 2 is the angle at the vertex $x^{(l)}$ in the triangle spanned by $x^{(i)}, x^{(j)}, x^{(l)}$ if $\{i,j,k,l\} = \{1,2,3,4\}$. In fact, if γ denotes this angle, $A := (a_{ij})_{i,j=1}^4 := \tilde{A}(\alpha)$, $C := (c_{ij})_{i,j=1}^4 := A^{-1}$, and

$$D := \begin{pmatrix} c_{ii} & c_{ij} & c_{il} \\ c_{ij} & c_{jj} & c_{jl} \\ c_{il} & c_{jl} & c_{ll} \end{pmatrix},$$

then

$$\cos(\gamma) = \frac{(D^{-1})_{12}}{\sqrt{(D^{-1})_{11}(D^{-1})_{22}}} = \frac{c_{il}c_{jl} - c_{ij}c_{ll}}{\sqrt{(c_{jj}c_{ll} - c_{jl}^2)(c_{ii}c_{ll} - c_{il}^2)}}$$
$$= \frac{a_{ik}a_{jk} - a_{ij}a_{kk}}{\sin(\alpha_{ik})\sin(\alpha_{jk})} = \frac{\cos(\alpha_{ij}) + \cos(\alpha_{ik})\cos(\alpha_{jk})}{\sin(\alpha_{ik})\sin(\alpha_{jk})} = \cos(\beta_{ij;k})$$

(comp. also [1, equation (4), p. 450; Engl.: p. 417]). Hence condition (iv) in Proposition 2 only requires what is true for every triangle in hyperbolic space, namely that the sum of the angles is less than π.

2. We next observe that Theorem 3 contains, as a special case, the well-known formula for the volume of a *totally asymptotic* or *ideal* tetrahedron, i.e. one all vertices of which lie at infinity. If $S := S(V)$ is such a tetrahedron, then $\alpha := \alpha(V)$ belongs to the set P_4' defined in (9), and formula (10) yields

$$(13) \qquad \text{volume of } S = \frac{1}{2}\,[\mathrm{Cl}_2(2\alpha_{12}) + \mathrm{Cl}_2(2\alpha_{13}) + \mathrm{Cl}_2(2\alpha_{23})].$$

This formula seems to have been stated first in [14, Lemma 2, p. 18] (comp. also [2, Proposition C.2.8, p. 100], [9, (14.52), p. 322], [17, Theorem 10.4.6, p. 470]). There the result is expressed in terms of the function $Л(x) = \frac{1}{2}\mathrm{Cl}_2(2x)$. We point out that this volume formula is stated erroneously in [16, p. 109].

3. Evidently, the dihedral angles are – up to renumbering of the vertices – the same for two isometric simplices. But also the converse is true, since, if $S = S(V)$ is not asymptotic, then the distances between its vertices $x^{(i)}$ are uniquely determined by α through

$$d(x^{(i)}, x^{(j)}) = \operatorname{arcosh}\left(\frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}\right), \quad (c_{ij})_{i,j=1}^{n} := \tilde{A}(\alpha)^{-1}.$$

(The asymptotic case follows therefrom by a limit argument.) Hence the quotient of P_n modulo the action of the permutation group operating on the set of vertices is in one-to-one correspondence with the set of isometry classes of $(n-1)$-dimensional simplices in hyperbolic space.

4. FORMULA FOR THE THREE-POINT FUNCTION IN EUCLIDEAN $\mathbb{R}^4$

We consider here the special case of the integral in (1) which corresponds to $m = 3$, $n = 4$. It can be written in the form

$$(14) \qquad J := \int_{\mathbb{R}^4} \prod_{k=1}^{3} (|q + b^{(k)}|^2 + \mu_k^2)^{-1}\, \mathrm{d}q_1 \cdots \mathrm{d}q_4$$

with $b^{(1)}, b^{(2)}, b^{(3)} \in \mathbb{R}^4$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{R}_+$. By translation and rotation invariance, J is a function of the nine parameters

$$(15) \qquad \mu_i, \qquad a_i := |b^{(k)} - b^{(j)}|, \qquad \gamma_i := \arccos\left(\frac{\langle b^{(j)} - b^{(i)}, b^{(k)} - b^{(i)}\rangle}{a_j a_k}\right),$$

where $\{i,j,k\} = \{1,2,3\}$. The latter six quantities are the Euclidean lengths and angles in the triangle with vertices $b^{(i)}$, $i = 1,2,3$, and hence are connected by three relations.

Theorem 4. *Let $\mu_i \in \mathbb{R}_+$, $b^{(i)} \in \mathbb{R}^4$, $i = 1,2,3$, and assume that the two vectors $b^{(2)} - b^{(1)}, b^{(3)} - b^{(1)}$ are linearly independent in $\mathbb{R}^4$. Define $J, a_i, \gamma_i, i = 1,2,3$, by* (14) *and* (15). *Let F denote twice the area of the triangle spanned by $b^{(i)}$, $i = 1,2,3$, in $\mathbb{R}^4$, i.e., $F = a_i a_j \sin(\gamma_k)$ if $\{i,j,k\} = \{1,2,3\}$. Furthermore, set*

$$u := a_1^2 a_2^2 a_3^2 + 2a_1a_2a_3 \sum_{i=1}^{3} \mu_i^2 a_i \cos(\gamma_i) + \sum_{i=1}^{3} a_i^2 \prod_{\substack{j=1\\ j\neq i}}^{3} (\mu_i^2 - \mu_j^2)$$

and, for $\{i,j,k\} = \{1,2,3\}$,

$$v_i := a_i a_j a_k \cos(\gamma_i) + \mu_i^2 a_i - \mu_j^2 a_j \cos(\gamma_k) - \mu_k^2 a_k \cos(\gamma_j), \qquad \delta_i := \arccos\left(\frac{v_i}{\sqrt{u}}\right).$$

Then

$$J = \frac{\pi^2}{2F}\Bigg\{\mathrm{Cl}_2(2\gamma_1) + \mathrm{Cl}_2(2\gamma_2) + \mathrm{Cl}_2(2\gamma_3) - \sum_{\substack{\{i,j,k\}=\{1,2,3\}\\ j<k}}$$
$$\times\,[\mathrm{Cl}_2(\pi+\gamma_i+\delta_j+\delta_k) + \mathrm{Cl}_2(\pi+\gamma_i+\delta_j-\delta_k)$$
$$+ \mathrm{Cl}_2(\pi+\gamma_i-\delta_j+\delta_k) + \mathrm{Cl}_2(\pi+\gamma_i-\delta_j-\delta_k)]\Bigg\}.$$

Proof. (a) From [16, Proposition 3 and the following remark, pp. 90, 94], we obtain

$$J = 4\pi^2 \int_{\mathbb{R}_+^4} \mathrm{e}^{-\langle Cx, x\rangle}\,\mathrm{d}x,$$

where

$$C = \begin{pmatrix} \mu_1^2 & (\mu_1^2+\mu_2^2+a_3^2)/2 & (\mu_1^2+\mu_3^2+a_2^2)/2 & 1 \\ (\mu_1^2+\mu_2^2+a_3^2)/2 & \mu_2^2 & (\mu_2^2+\mu_3^2+a_1^2)/2 & 1 \\ (\mu_1^2+\mu_3^2+a_2^2)/2 & (\mu_2^2+\mu_3^2+a_1^2)/2 & \mu_3^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Upon setting $C = (c_{ij})_{i,j=1}^4$, elementary calculations yield, for $\{i,j,k\} = \{1,2,3\}$,

$$C_{ii}^{\mathrm{ad}} = \det\begin{pmatrix} c_{jj} & c_{jk} & 1 \\ c_{jk} & c_{kk} & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2c_{jk} - c_{jj} - c_{kk} = a_i^2 \tag{16}$$

and

$$\det C = -\det\begin{pmatrix} c_{12}-c_{11} & c_{13}-c_{11} & 1 \\ c_{22}-c_{12} & c_{23}-c_{12} & 1 \\ c_{23}-c_{13} & c_{33}-c_{13} & 1 \end{pmatrix}$$
$$= -\det\begin{pmatrix} c_{22}-2c_{12}+c_{11} & c_{23}-c_{12}-c_{13}+c_{11} \\ c_{23}-c_{12}-c_{13}+c_{11} & c_{33}-2c_{13}+c_{11} \end{pmatrix}$$
$$= -\det\begin{pmatrix} -a_3^2 & (a_1^2-a_2^2-a_3^2)/2 \\ (a_1^2-a_2^2-a_3^2)/2 & -a_2^2 \end{pmatrix}$$
$$= -\det\begin{pmatrix} a_3^2 & a_2a_3\cos(\gamma_1) \\ a_2a_3\cos(\gamma_1) & a_2^2 \end{pmatrix} = -a_2^2a_3^2\sin^2(\gamma_1) = -F^2.$$

Therefore, the sequence

$$1,\ c_{44}=0,\ c_{33}c_{44}-c_{34}^2=-1,\ C_{11}^{\mathrm{ad}}=a_1^2,\ \det C = -F^2$$

contains exactly three sign changes and hence $C \in Q_4$ and Theorem 2 can be applied.

(b) Similarly as in (16), we have, for $\{i,j,k\} = \{1,2,3\}$,

$$C_{ij}^{\mathrm{ad}} = -\det\begin{pmatrix} c_{ij} & c_{ik} & 1 \\ c_{jk} & c_{kk} & 1 \\ 1 & 1 & 0 \end{pmatrix} = c_{kk} + c_{ij} - c_{ik} - c_{jk} = -a_ia_j\cos(\gamma_k). \tag{17}$$

If, according to Proposition 1, $\alpha \in P_4$ and $d \in\,]0,\infty[^4$ are determined by the equation $C = h_4(\alpha, d)$, then (16) and (17) imply, for $i < j$,

$$\alpha_{ij} = \arccos\left(\frac{-C_{ij}^{\mathrm{ad}}}{\sqrt{C_{ii}^{\mathrm{ad}} C_{jj}^{\mathrm{ad}}}}\right) = \gamma_k.$$

A lengthier, but elementary calculation furnishes (still for $\{i,j,k\} = \{1,2,3\}$ and with u, v_i as in the theorem)

$$C_{44}^{\mathrm{ad}} = \frac{u}{4}, \qquad C_{i4}^{\mathrm{ad}} = \frac{a_i}{2}\left[\mu_j^2 a_j \cos(\gamma_k) + \mu_k^2 a_k \cos(\gamma_j) - \mu_i^2 a_i - a_i a_j a_k \cos(\gamma_i)\right]$$

and hence

$$\alpha_{i4} = \arccos\left(\frac{-C_{i4}^{\mathrm{ad}}}{\sqrt{C_{ii}^{\mathrm{ad}} C_{44}^{\mathrm{ad}}}}\right) = \arccos\left(\frac{v_i}{\sqrt{u}}\right) = \delta_i.$$

Application of equation (10) then completes the proof. □

Remark. The special case of an ideal tetrahedron (cf. Remark 2 to Theorem 3) occurs if all the masses μ_i, $i = 1,2,3$, vanish. Then $\delta_i = \gamma_i$, $i = 1,2,3$, and thus one obtains (comp. (13))

$$J = \frac{\pi^2}{F}\left[\mathrm{Cl}_2(2\gamma_1) + \mathrm{Cl}_2(2\gamma_2) + \mathrm{Cl}_2(2\gamma_2)\right].$$

Notice that FJ, in this case, does not depend on the quantities a_i, $i = 1,2,3$.

We finally mention that Theorem 2 also yields a representation of the four-point function if at least one of the particles is massless.

Note added in proof. As the author found out only after completion of the proof sheets, formula (10) has been published already in 1988 by E.B. Vinberg in the Proc. 1st Siberian Winter School on Algebra and Analysis; Engl. Transl. in AMS Translations, Ser. 2, vol. 148 (1991), formula (24), p. 25.

Vinberg's proof of formula (10) is based on a geometric dissection method and hence is different from the above more analytic one.

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